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DIFFUSION OF A PASSIVE SCALAR

Quarterly Technical Report

by

D. R. S. Ko and I. E. Alber

1972 February

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13 ABSTRACT

An analytical model for the prediction of mixing and decay of a passive scalar within the turbulent wake of a submerged body operating beneath the ocean surface. The scaling parameters are identified from the governing equations. Moreover, a review of the existing theoretical and experimental studies of diffusion in an ocean environment is given. The application to the present problem of interest is briefly discussed.

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SUMMARY

The main objective of this study is to develop an analytical model for the prediction of the mixing and decay of a passive scalar within the turbulent wake of a submarine operating beneath the ocean surface. Even though the interest is on the passive scalar, the essence of the problem is the determination of the dynamics of a submarine wake. Once the wake flow field is determined, the decay of the concentration of a passive scalar can then be easily obtained.

Depending on the distance behind the submarine, two somewhat different technical problems are encountered in the analytical modeling of the wake. For the region relatively close to the submarine, (wake age of the order of an hour), the flow field is influenced by the effects directly caused by the body. The proper analytical model must take into consideration the following characteristics of a submarine wake: the turbulence generated by the body and the propeller; the momentumless wake behind a submarine moving at a constant speed, and the stratified nature of the fluid media. For the region far behind the submarine, the sub-generated perturbations decay below the natural background turbulence levels. Further diffusion of the passive scalar in this region of the wake is expected to be dominated by the ocean background turbulence and the character of the diffusion plume.

An analytical model for the near wake region, based on the concepts of local similarity and turbulent entrainment of the external fluid, is formulated in Section 2. A turbulent model is introduced which pays particular attention to the effect of stratification. Using basically an integral conservation approach, the problem is reduced to solving a set of five ordinary differential equations with given initial conditions. By properly normalizing the governing equations, two primary scaling parameters for the submarine wake are obtained. The most important scaling parameter is the turbulent Froude number $F_r = u'_{mo}/NL$, where u'_{mo} denotes the initial turbulent intensity, $N = (-\frac{g}{\rho}\frac{d\rho}{dz})^{1/2}$ the Brunt-Vasaila frequency and L is the initial wake width. To a lesser degree, the initial turbulent intensity level u'_{mo}/U_o provides another governing parameter.

Theoretical or empirical estimates of the value of these parameters, for any particular experiment or field condition is not available at the present time. However, a parametric study will soon be performed and comparisons with existing laboratory experiments will be attempted. These results will be discussed in the final report. Meanwhile, effort, both analytically and experimentally, should be directed toward a logical and consistent determination of the value of these governing parameters.

The effects of the background turbulent ocean dynamics has been completely ignored in the formulation presented in Section 2, which is valid for the relatively near wake region. As a first step toward increasing the understanding of this important effect, an in-depth review is presented in Section 3 of existing theoretical and experimental studies of diffusion in an ocean environment.

In analytically modeling the diffusion of a passive scalar, the diffusion coefficients K_i must be known for both vertical and horizontal transport. The review in Section 3, concludes that horizontal diffusion in the ocean dominates over vertical diffusion ($K_h > 10^4 K_v$) because of the effect of the stratified media in retarding vertical turbulence fluctuations. Thus diffusion from a moving continuous source proceeds as a thin fan in the opposite direction from the source trajectory. The horizontal diffusion coefficient is estimated with the aid of the theories of Taylor, Batchelor and Richardson, showing that K_h is proportional to the 4/3 power of the scale of diffusion ℓ. Examination of a number of dye diffusion experiments in the ocean indicates that K_h is approximately represented by the Richardson 4/3 law, ($K \sim \ell^{1.15}$) and that the lateral dimension σ of the dye plume increases rapidly with time $\sigma \sim t^{1.17}$ ($\sigma \sim t^{3/2}$ for the Richardson law). This rate of growth indicates, for example, that the size of a dye patch would be on the order of several kilometers after a period of diffusion of one day.

In Section 3, the diffusion coefficients and rate of spread are used to estimate the concentration decay behind a continuous source. For a point source, it is shown that the concentration c falls off inversely as the diffusion time to the 1.17 power (or 3/2 power for the Richardson law). Far field solutions are also presented for diffusion

from a finite bounded concentration field. Such initial conditions can be supplied by the near field analysis of Section 2 where sub-generated turbulence is important. Further numerical evaluations of the concentration distributions for typical ocean and sub conditions will be presented in the final report of this study.

CONTENTS

ILLUSTRATIONS

1. INTRODUCTION

The main objective of this study is to develop an analytical model for the prediction of the mixing and decay of a passive scalar within the turbulent wake of a submarine operating beneath the ocean surface. Even though the interest is on the passive scalar, the essence of the problem is the determination of the dynamics of a submarine wake. The decay of the concentration of a passive scalar could then be easily obtained from the conservation equation for a known flow field.

Numerous theoretical and experimental studies on the wakes behind unpropelled bodies in a uniform medium have been reported. The knowledge acquired through these studies is not readily applicable to the present case of interest because of the following two characteristics of a submarine wake:

1) For a submarine moving at a constant velocity, there is no net axial momentum being deposited in the wake. The phenomena associated with these momentumless wakes are, in many respects, different from a wake having a momentum deficit.

2) The ocean environment is generally stably stratified. Because of the turbulent mixing in the wake, fluid particles within the wake are out of equilibrium with the stratified environment. This results in an increase of the potential energy of the wake. Gravitational restoring forces continuously convert part of this potential energy into kinetic energy and the wake tends to collapse vertically and spread out horizontally.

These characteristics of a submarine wake have not been subjected to any rigorous theoretical treatment to date. In a recent report, Ko[1] has presented a simplified phenomenological model with the main emphasis on identifying the important physical mechanisms which affect the wake dynamics. Even though a fairly good agreement with some existing laboratory data have been obtained, this simplified model must be improved before it can be used as a reliable prediction tool. This simplified model along with several improvements is described in Section 2 of this report.

Another related problem having to do with the effect of the ocean background turbulence on the wake diffusion has never been addressed previously. Based on the previous calculations (see Ko[1]), the turbulence

generated by the submarine and its propulsion system persists only for a relatively short distance because of the stabilizing effect of the stratified environment. Therefore, we might expect the far wake diffusion to be dominated by the background turbulence. However, our present level of understanding in this regard is rather limited. As a first step, we will present in Section 3 a partial survey of the existing understanding of the oceanic turbulent diffusion. The application to the submarine wake will also be indicated.

2. SUBMARINE WAKE MODEL

The basic approach follows closely the analysis of Reference 1 with a few improvements. Some of the assumptions in the previous analysis will be relaxed in order to achieve a more reliable first-order predictive tool, at least for the case of a negligible background turbulent diffusion. Therefore, most of the details in the derivation of the governing equations will be omitted here and reference may be made to the previous analysis.

For a submarine moving with a uniform speed U_0 at a depth H below the ocean surface, a Cartesian sub-fixed coordinate system is chosen as shown in Figure 1. The undisturbed medium outside the wake is assumed to be horizontally stratified with the density given by

$$\rho_0 = \bar{\rho}(1 - \alpha z) \tag{2-1}$$

with $\bar{\rho}$ denoting the mean density at the submarine depth and α being the given constant gradient. The density field inside the wake is taken to be

$$\rho_i = \bar{\rho}(1 - \beta z) \tag{2-2}$$

with β, the internal density gradient, reflecting the degree of mixing inside the wake. The effects of density discontinuity that exists at the wake boundary for this assumed density distribution will be considered in a later study. One of the main improvements of the present modeling will be the inclusion of a suitable mixing equation to account for the variation of β.

The concepts of local similarity and turbulent entrainment of the external fluid will be used as before. Assuming that there is no direct effect of turbulence on the return of a displaced fluid particle inside the wake to its equilibrium position, the following governing equation is obtained from the momentum equations

$$U_0 \left(1 + \frac{b^2}{a^2}\right) \frac{df_1}{dx} = \frac{b^2}{a^2} (\alpha - \beta)g - \left(1 - \frac{b^2}{a^2}\right) f_1{}^2 \tag{2-3}$$

where f_1 is the local velocity scaling factor such that

$$v = yf_1 \text{ and } w = -zf_1 \tag{2-4}$$

The values of a and b represent the major and minor axis of the elliptical wake boundary given by

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{2-5}$$

The governing equations for a and b are given by

$$\begin{aligned} U_o \frac{da}{dx} &= af_1 + E_y \\ U_o \frac{db}{dx} &= -af_1 + E_z \end{aligned} \tag{2-6}$$

with E_y and E_z denoting the turbulent entrainment velocities in the y and z directions at the wake boundary y = a and z = b. If the entrainment velocity is assumed to be proportional to the distance from the center of the wake, then we have

$$E_z = \frac{b}{a} E_y = \frac{b}{2} \frac{U_o}{A} \frac{dA}{dX} \tag{2-7}$$

where $A = \pi ab$ denotes the area of the wake. The entrainment velocity is then related to the local turbulent intensity and scale using the experimental results of Naudascher, which gives

$$\frac{U_o}{A} \frac{dA}{dX} = \frac{K_1 u_m'}{a} \tag{2-8}$$

with $K_1 \approx 2.8$.

The formulation presented thus far is identical to that given in Reference 1. The intent of what follows here is to more accurately account for the effect of density stratification on the rate-of-change of the turbulent energy and its effect on the wake density gradient β.

The total integrated turbulent energy in the wake is given by

$$E_t = \bar{\rho}\, u_m'^2 A \tag{2-9}$$

where u_m' is the averaged turbulent velocity over the wake. An equation for the rate-of-change of the turbulent energy within the wake in a stratified medium can be written as

$$U_o \frac{dE_t}{dx} = P - D - S \tag{2-10}$$

where P = rate of production

D = rate of dissipation

S = stratification effect on turbulent energy exchange

The modeling of the production and the dissipation terms follows the same line of reasoning as Reference 1 in which it is shown

$$P = K_2\, \pi \bar{\rho} a \frac{u_m'^5}{U_o^2} \tag{2-11}$$

and

$$D = C \pi \bar{\rho}\, u_m'^3\, a \tag{2-12}$$

The constants are determined from the experiment of Naudascher to be $K_2 = 140$ and $C = 8$. The modeling for the stratification effect on the turbulent energy balance marks the main difference of the present analysis from the previous one. The previous analysis was based on an overall potential and kinetic energy conservation concept which led to a representation of the term

$$\iint \overline{\rho' w'}\, g\, dA$$

which appears in the turbulent energy equation. However, the fact that the effect of stratification on turbulence vanishes when $\beta \equiv \alpha$ leaves some doubt as to its validity. Furthermore, because of the need to model the $(\overline{\rho' w'})$ term for the mixing equation, there is little reason not to model the effect of stratification on the turbulent kinetic energy directly.

Since this modeling was not considered in the previous report, we will go into some detail in its formulation here.

The modeling is based on the concepts of similarity and eddy diffusivity. Let's assume

$$\overline{\rho' w'} = (\overline{\rho' w'})_{max} \, F_c(y^*, z^*) \tag{2-13}$$

where $y^* = y/a$, $z^* = z/b$, and F_c is a distribution function which vanishes at the wake boundary and has a maximum at the center of the wake. One simple form of F_c can be taken as

$$F_c = 1 - y^{*2} - z^{*2} \tag{2-14}$$

Using an eddy diffusivity formulation, we let

$$\begin{aligned}(\overline{\rho' w'})_{max} &= -\varepsilon \frac{\partial \rho_i}{\partial z} \\ &= \varepsilon \, \beta \, \overline{\rho} \\ &= K' u_m' b \beta \overline{\rho}\end{aligned} \tag{2-15}$$

The value of K' can be estimated from the diffusivity for an incompressible turbulent wake (e.g., Townsend) to be about 0.2. With the above assumptions, evaluation of the term S is

$$\begin{aligned}S = g \int_A \int \overline{\rho' w'} \, dA &= 4 \, g \, K' u_m' \beta b \overline{\rho} \int_0^b dz \int_0^{a\sqrt{1 - z^{*2}}} (1 - y^{*2} - z^{*2}) dy \\ &= \frac{\pi}{2} K' u_m' \beta g \overline{\rho} \, ab^2\end{aligned} \tag{2-16}$$

Therefore, the turbulent energy equation is then

$$U_0 \frac{d(u_m'^2 ab)}{dx} = K_2 a \frac{u_m'^5}{U_0^2} - C u_m'^3 a - \frac{K'}{2} \beta a b^2 g u_m' \tag{2-17}$$

The proper mixing model for determining the local density gradient within the turbulent wake is obtained from the averaged density conservation equation, which gives

$$U_0 \frac{\partial \rho_i}{\partial x} + \overline{v} \frac{\partial \rho_i}{} + \overline{w} \frac{\partial \rho_i}{\partial z} = - \left[\frac{\partial}{\partial x} \overline{\rho' u'} + \frac{\partial}{\partial y} \overline{\rho' v'} + \frac{\partial}{\partial z} \overline{\rho' w'} \right] \qquad (2\text{-}18)$$

Assuming $\frac{\partial}{\partial x} \overline{\rho' u'}$ is small and noting that

$$\rho_i = \overline{\rho}(1 - \beta z), \ \overline{v} = yf_1, \ \overline{w} = -zf_1$$

an integration of the LHS of equation (2-18) over the upper half of the wake gives

$$\iint (U_0 \frac{\partial \rho_i}{\partial x} + \overline{v} \frac{\partial \rho_i}{\partial y} + \overline{w} \frac{\partial \rho_i}{\partial z}) dA = \frac{2}{3} \overline{\rho} \Big[(\alpha^* - \beta^*) \ U_0 \frac{dab}{dx} + \alpha^* a E_z$$

$$- U_0 ab \frac{d\beta^*}{\partial x} \Big] \qquad (2\text{-}19)$$

where $\alpha^* = \alpha b$ and $\beta^* = \beta b$. The integration of the RHS of equation (2-18) results in only one non-vanishing term

$$T = \int_a^a \overline{\rho' w'} \ (z = 0) \ dy \qquad (2\text{-}20)$$

which represents the transport of mass across the interface z = 0. With equations (2-13) - (2-15), this transport term can be readily evaluated to be

$$T = \frac{4}{3} K' u_m' \ \overline{\rho} \ \beta ab \qquad (2\text{-}21)$$

Then, by equating (2-19) and (2-21), we obtain the equation for mixing

$$(\alpha^* - \beta^*) \frac{U_0}{ab} \frac{dab}{dx} + \alpha^* \frac{E_z}{b} - U_0 \frac{d\beta^*}{dx} = 2K' u_m' \beta \qquad (2\text{-}22)$$

2.1 NON-DIMENSIONAL GOVERNING EQUATIONS

By using the following characteristic quantities

Length ... L (characteristic dimension of the flow)

Velocity ... U_o or $U_{ref} = u'_{mo}$ (initial turbulent intensity)

Time ... N^{-1} ($N = \sqrt{\alpha g}$ .. Vaisala-Brunt frequency)

and let

$$a^* = a/L,\ b^* = b/L$$

$$u_m^* = u_m'/U_{ref},\ E_z^* = E_z/U_{ref},\ E_y^* = E_y/U_{ref}$$
$$f_1^* = f_1/N,\ X^* = XN/U_o$$

and define two non-dimensional numbers

$$F_r = \frac{U_{ref}}{NL} \text{ ... turbulent Froude number}$$

$$U_N = U_{ref}/U_o$$

Then, the governing equations can be summarized as

$$\frac{da^*}{dx^*} = a^*f_1^* + F_rE_y^*$$

$$\frac{db^*}{dx^*} = -b^*f_1^* + F_rE_z^*$$

with $E_z^* = \frac{K_1}{2} u_m^* \frac{b^*}{a^*}$

$$E_y^* = \frac{K_1}{2} u_m^*$$

$$\left(1 + \frac{b^{*2}}{a^{*2}}\right) \frac{df_1^*}{dx^*} = \left(1 - \frac{\beta^*}{\alpha^*}\right) \frac{b^{*2}}{a^{*2}} - \left(1 - \frac{b^{*2}}{a^{*2}}\right) f_1^{*2}$$

$$\frac{d}{dx^*} u_m^{*2} = F_r\left[U_N^2K_2 \frac{u_m^{*5}}{b^*} - C\frac{u_m^{*3}}{b^*} - K_1\frac{u_m^{*3}}{a^*} - \frac{1}{F_r^2}\frac{K'}{2}\frac{\beta^*}{\alpha^*} u_m^*b^*\right]$$

$$\frac{d\beta^*}{dx^*} = F_r \frac{u_m^*}{b^*}\left[\frac{3}{2}\frac{b^*}{a^*}K_1\alpha^* - (2K' + K_1 \frac{b^*}{a^*})\ \beta^*\right]$$

The fact that the turbulent energy equation allows a non-vanishing derivative of u_m^* when $u_m^* = 0$ requires a constraint to the numerical program such that u_m^* is set to be identically zero once it reaches zero. Another somewhat artificial constraint is placed on the magnitude of β such that it will not exceed α.

Now assuming the wake to be initially circular and taking the characteristic length L to be equal to the initial wake radius, the proper set of the initial conditions is

$$a_o^* = b_o^* = 1$$

$$f_{1o}^* = 0$$

$$u_{mo}^* = 1$$

and a specified β_o^*.

The equations point out that the proper scaling parameters for the submarine wake are the turbulent Froude number $F_r = \frac{u'_{mo}}{NL}$ and the initial turbulent intensity level $U_N = u'_{mo}/U_o$. In addition, the initial degree of mixing, represented by β_o/α appearing in the form of an initial condition, constitutes an additional parameter in the problem. The value of these parameters for any particular experiment or field condition is not apparent at the present time. Even though these parameters are not completely independent of one another, the actual determination of their magnitudes will have to rely quite heavily on empirical results. While searching for the proper way of choosing these parameters, parametric studies will be performed and the results will be presented in the final report.

3. PRELIMINARY CONSIDERATIONS OF TURBULENT DIFFUSION OF A PASSIVE SCALAR IN THE OCEAN

To describe the diffusion of a passive scalar (e.g., a radioactive trace contaminant) in the ocean, one has to quantitatively understand the kinematics of ocean turbulence. If one seeks to find the mean concentration of a contaminant, $\overline{c}_i(x,y,z)$, diffusing from a fixed source in an ocean flow field, the usual approach is to seek solutions of the time averaged species conservation equation

$$\frac{D\overline{c}_i}{Dt} = - \frac{\partial}{\partial x_j} (\overline{c_i' u_j'}) + \overline{\dot{\omega}_i} \qquad (3\text{-}1)$$

where $\overline{c_i' u_j'}$ represents the species flux in the j direction and $\overline{\dot{\omega}_i}$ represents the time averaged rate of production (or loss) of species i by chemical reaction or radioactive decay. $\frac{D}{Dt}$ represents a material derivative.

The derivation of Eq. (3-1) assumes that the nature of the flow field is described by a stationary random function of time. That is, when the turbulence properties are averaged over a suitable time increment T, a stationary value for the turbulence properties is obtained. In a meandering ocean flow field, various averaged turbulence properties may result depending on the magnitude of the time scale T. For example, large scale horizontal ocean diffusion properties only achieve a stationary value when averages are taken over periods of several hours to a day.

For the analysis to be presented in this report, stationary values of the turbulence properties will be assumed in order to provide a basic estimate of the diffusional characteristics of oceanic turbulence.

3.1 DIFFUSION COEFFICIENTS

To solve the species conservation equation, Eq. (3-1), expressions for the species flux $\overline{c_i' u_j'}$ must be adopted. Using the Boussinesq approximation, turbulent diffusion coefficients can be defined by the equation

$$K_j = - \overline{c_i' u_j'}/(\partial \overline{c}_i/\partial x_j)$$

These coefficients do not necessarily assume a constant value and may be functions of the distance x from a source (or time $t = x/U$). Specific forms $K = K(x)$ have been suggested by Sutton[2] for various stability conditions corresponding to the atmospheric diffusion of smoke plumes. The change of the diffusion constant with time is best calculated by Taylor's[3] diffusion theory of continuous movements in a homogeneous turbulent field.

Taylor defined the function $R(\tau)$ as the coefficient of correlation between the turbulent velocity u_t of a particle of fluid at time t and the velocity $u_{t+\tau}$ of the same particle at a time τ later, such that,

$$R(\tau) = \overline{u_t u_{t+\tau}}/\overline{u^2} \tag{3-3}$$

where $\overline{u^2}$ represents the mean square velocity fluctuation intensity at time t. If ξ is the displacement of a particle of fluid at time t from its original position at time $t = 0$, Taylor proved that

$$\overline{\xi^2} \equiv \sigma^2 = 2\,\overline{u^2} \int_0^t \int_0^{t'} R(\tau)\,d\tau\,dt' \tag{3-4}$$

For small values of t, such that $R(\tau) = 1.0$

$$\overline{\xi^2} = \sigma^2 = \overline{u^2}t^2 \tag{3-5}$$

For times $t \gg t^*$, (where t^* denotes the Lagrangian time scale $= \int_0^\infty R(\tau)d\tau$) the correlation $R(\tau)$ is effectively zero. Hence if diffusion proceeds for times much greater than t^*, the integral in Eq. (3-4) will reach a limiting value I. In that case, the mean square displacement is proportional to the time of diffusion, i.e.,

$$\sigma^2 = 2\,\overline{u^2}\,It$$

or

$$\sigma^2 = 2\sqrt{\overline{u^2}}\,\Lambda_L t \tag{3-6}$$

where $\Lambda_L = \sqrt{\overline{u^2}} \int_0^{t'} R(\tau)d\tau$ is defined as the Lagrangian integral length. If the particles had been dispersed by a diffusion process obeying the diffusion equation

$$\frac{\partial c}{\partial t} = K_y \frac{\partial^2 c}{\partial_y^2} \tag{3-7}$$

the solution of Eq. (3-7) for the concentration c would be

$$c = \frac{\text{const}}{\sqrt{t}} \exp\left(-y^2/4K_y t\right) \tag{3-8a}$$

or

$$c = \frac{\text{const}}{\sigma_y} \exp\left[-\frac{1}{2}\frac{y^2}{\sigma_y^2}\right] \tag{3-8b}$$

Equations (3-8a) and (3-8b) yield a mean square displacement σ_y of fluid particles in the y dimension; given by

$$\sigma_y^2 = 2\,K_y t \tag{3-9}$$

Based on this relation, one can define a diffusion coefficient from Eq. (3-9) in terms of the mean square displacement, i.e.,

$$K_y = \frac{1}{2}\frac{d\sigma_y^2}{dt} \tag{3-10}$$

The diffusion coefficient K_y given by Eq. (3-10) can thus be used directly in a simple diffusion equation of the form given by Eq. (3-7).

Batchelor[4] extended Taylor's ideas with the aid of dimensional arguments. He obtained expressions for the diffusion coefficients K (or $d\sigma^2/dt$) applicable to the horizontal diffusion of a passive scalar (or trace contaminant) in a homogeneous ocean. The characteristic velocity scale is related to the rate of turbulent energy dissipation per unit mass ε, and to the standard deviation of the initial source field σ_0 by the expression

$$\sqrt{\overline{u^2}} \sim (\varepsilon c_0)^{1/3} \tag{3-11}$$

Batchelor finds three regimes of relative diffusion, the two given by Taylor's relations [Eq. (3-5) and (3-6)] and one important intermediate regime given by the following rates of growth

$$K = \frac{1}{2}\frac{d\sigma^2}{dt} = C_1 t(\varepsilon\sigma_0)^{2/3} \text{ (initial) } t \ll t^* \tag{3-12a}$$

$$K = \frac{1}{2}\frac{d\sigma^2}{dt} = C_2 \varepsilon (t - t_1)^2 \text{ (intermediate) } t \sim t^* \tag{3-12b}$$

with $t_1 = C_3 \sigma_0^{2/3} \varepsilon^{-1/3}$

$$K = \frac{1}{2}\frac{d\sigma^2}{dt} = C_4 \sqrt{\overline{u^2}}\, \Lambda_L \text{ (asymptotic) } t \gg t^* \tag{3-12c}$$

For the intermediate times as given by Eq. (3-12b)

$$K \sim \varepsilon t^2 \sim \varepsilon^{1/3} \sigma^{4/3} \tag{3-13}$$

and

$$\sigma^2 \sim \varepsilon t^3 \tag{3-14}$$

Thus the diffusivity grows as the 4/3 power of the particle field (or plume) size, and the plume grows as the 3/2 power of the time. The 4/3 law was first derived by Richardson(5) in 1926. Richardson's law of relative diffusion indicates that diffusion is an accelerating process, the rate of growth increasing with the size of the field, at least while the supply of eddies of the requisite size lasts.

3.2 TWO DIMENSIONAL HORIZONTAL DIFFUSION

Richardson's law has been found to describe fairly well the horizontal diffusion in the ocean. Experimentally, horizontal diffusion has been studied extensively using the fluorescent dye technique. An overall discussion of such experiments is presented in the review article of Bowden(6). In some cases the dye has been released continuously from a point source into a current and the dispersion of the plume observed. In other cases the dye has been released as instantaneously as possible and the dispersion of the resulting patch observed as a function of horizontal coordinates and time. In both cases the initial dispersion is usually three dimensional but after a relatively short time the vertical dispersion becomes severely restricted by the presence of a thermocline, and the subsequent spreading is effectively two dimensional. In that case, the diffusion equation for a point source in a uniform turbulent flow of velocity U, assumes the form

$$U \frac{\partial c_i}{\partial x} = K_y \frac{\partial^2 c_i}{\partial y^2} \tag{3-15}$$

For the case where an isolated source emits a mass per unit time of species i at a constant rate Q_i^* per unit distance in the vertical direction, Eq. (3-15) has the solution

$$c_i = \frac{Q_i^*/\rho}{\sqrt{2\pi}\sigma_y U} \exp\left[-\frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2\right] \tag{3-16}$$

where

$$\frac{1}{2}\frac{d\sigma_y^{\,2}}{dt} = K_y \tag{3-17}$$

Okubo[7] in 1968 has made a review of 20 sets of data on instantaneous releases of dye in the ocean over the last ten years. They include experiments in the North Sea and in the Cape Kennedy area, and off of Southern California. Okubo has tabulated the variance σ_{rc}^2, and the apparent diffusivity K_a as a function of time t for each of the 20 releases. These results are shown in Figures 2 and 3 where σ_{rc}^2 is shown plotted against the diffusion time and K_a is plotted against the scale of diffusion, $\ell = 3\sigma_{rc}$. These diagrams cover time scales ranging from 1 hr. to 1 month and length scales from 100m to 100km. For the increase of variance with time, Okubo found that a best fit of the data could be obtained if

$$\sigma_{rc}^2 = .0108t^{2.34} \tag{3-18}$$

This is a somewhat smaller rate of growth than given by the t^3 law based on the Richardson theory, Eq. (3-14). The relation between the apparent diffusivity and scale was found by Okubo to be

$$K_a = .01\ell^{1.15} \tag{3-19a}$$

compared with the theoretical $\ell^{4/3}$ law. However, the differences with the Richardson theory are not large.

If one assumes that the horizontal diffusivity can be written in the form

$$K_a = a\ell^n$$

and that (3-19b)

$$\ell = b\sigma$$

then by Eq. (3-10) one can readily show that

$$\sigma_y = \left[(2 - n)\tilde{a}t\right]^{1/(2-n)} \tag{3-20}$$

where

$$\tilde{a} = b^n a$$

The concentration distribution of a passive scalar downstream of a continuous point source, moving at a velocity U, in a quiescent ocean, will be given by Eqs. (3-16) and (3-20), i.e.,

$$\overline{c_i} = \frac{Q_i{}^*/\rho U}{\sqrt{2\pi}\left[(2 - n)\tilde{a}x/U\right]^{1/(2-n)}} \exp\left[- \frac{y^2/2}{\left[(2 - n)\tilde{a}x/U\right]^{2/(2-n)}}\right] \tag{3-21}$$

For the constants n = 1.15 the concentration drops off rapidly with distance from the source x as

$$\overline{c_i} \sim \frac{1}{x^{1.17}} \tag{3-22}$$

and the size of the field increases as

$$\sigma_y \sim x^{1.17} \tag{3-23}$$

If the Richardson's 4/3 law is assumed to be valid, n = 4/3, hence

$$\overline{c_i} \sim \frac{1}{x^{3/2}}\ ,\ \sigma_y \sim x^{1.5} \tag{3-24}$$

Calculations of concentration versus distance behind the source are presented in Fig. 4 for both the n = 1.15 and n = 4/3 laws.

In the following section more general solutions are presented of the diffusion equation including the effects of vertical diffusion, and radioactive decay. In addition solutions will be presented which allow for arbitrary initial conditions based on the near field submarine wake solutions of Ko[1] in a stratified medium. Diffusion calculations for typical submarine wakes will be included in the final report of this study.

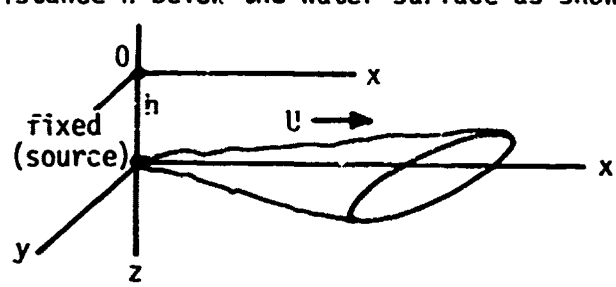
3.3 THREE DIMENSIONAL DIFFUSION WITH DECAY

The semi-empirical steady state diffusion equation is written below for a three dimensional Cartesian coordinate system x, y, z with stream velocity U in the x direction

$$U \frac{\partial \overline{c_i}}{\partial x} = \frac{\partial}{\partial y} \left(K_y \frac{\partial \overline{c_i}}{\partial y}\right) + \frac{\partial}{\partial z} \left(K_z \frac{\partial \overline{c_i}}{\partial z}\right) - \frac{\overline{c_i}}{\tau_c} \tag{3-25}$$

Here the z axis is directed vertically downward, K_y and K_z are the turbulent diffusivities defined by Eq. (3-2) in the y and z directions.* To take into account a possible exponential decrease of $\overline{c_i}$ with characteristic decay time τ_c (e.g., caused by radioactive decay), the last term on the right of Eq. (3-25) has been chosen to represent the production term, $\overline{\dot{\omega}_i}$, in Eq. (3-1).

A special solution of the diffusion equation (3-25) is possible for an isolated fixed source of strength Q_i (mass per unit time) located at (0, 0, h) [a distance h below the water surface as shown below]



*A term of the form $\frac{\partial}{\partial x} (K_x \frac{\partial \overline{c_i}}{\partial x})$ is neglected in Eq. (3-25) in comparison to $U \frac{\partial \overline{c_i}}{\partial x}$. This assumption is quite good for problems involving continuous sources.

For the case of a uniform stream velocity U, and for values of K_y and K_z, which are constants or functions of x, the solution of Eq. (3-25) is found to be[8]

$$\overline{c_i}\ (x,\ y,\ z) = \frac{Q_i/\rho}{2\pi\sigma_y\sigma_z U} \exp\left[-\frac{x}{U\tau_c}\right] \exp\left[-\frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2\right]$$

$$\bullet \left\{\exp\left[-\frac{1}{2}\left(\frac{z-h}{\sigma_z}\right)^2\right] + \exp\left[-\frac{1}{2}\left(\frac{z+h}{\sigma_z}\right)^2\right]\right\} \quad (3\text{-}26)$$

The dispersion coefficients are $\sigma_y = \left(\frac{2K_y x}{U}\right)^{1/2}$ and $\sigma_z = \left(\frac{2K_z x}{U}\right)^{1/2}$ (3-27)

This solution corresponds to the following initial and boundary conditions imposed on Eq. (3-25).

$$x = 0: \quad \overline{c_i} \rightarrow Q_i \frac{\delta(y)\ \delta(h - z)}{\rho U}$$

$$z = 0: \quad \frac{\partial \overline{c_i}}{\partial z} = 0 \quad \text{(non-catalytic surface)}$$

$$z \rightarrow \infty: \quad \overline{c_i} \rightarrow 0$$

If the source is far below the surface h >> σ_z, then Eq. (3-26) takes the simple Gaussian form

$$\overline{c_i} = \frac{Q_i/\rho}{2\pi\sigma_y\sigma_z U} \exp\left[-\frac{x}{U\tau_c}\right] \exp\left[-\frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2 - \frac{1}{2}\left(\frac{\tilde{z}}{\sigma_z}\right)^2\right] \quad (3\text{-}28)$$

$$h \gg \sigma_z$$

where $\tilde{z} = z - h$.

3.4 VERTICAL DIFFUSION

In order to make quantitative use of Eq. (3-26), in the calculation of three dimensional concentration distributions, the diffusion coefficients K_y, and K_z must be given or the dependence of σ_y and σ_z on the distance x must be specified. In the previous section it was shown that the horizontal

diffusion coefficient K_y in the ocean could be characterized by the Richardson 4/3 law or a closely related empirical expression of the form $K_y = a\ell^n$. Typical ocean measurements (see Fig. 3) yield values of K_y in the range

$$K_y = 0(10^3 \text{ to } 10^7)\ \frac{cm^2}{sec}$$

Corresponding values of the vertical diffusion coefficient K_z are usually much lower and are reported to lie in the range[9]

$$K_z = 0(1 \text{ to } 10^3)\ \frac{cm^2}{sec}$$

The principal mechanism by which turbulence is generated in the upper layers of the ocean is by the action of the wind shear acting on the surface layer. From observational data, Sverdrup[10] proposed the following empirical relationship between N_{z0} [the surface vertical turbulent eddy stress coefficient = $-\overline{u'w'}/(\partial\bar{u}/\partial z)$] and the wind speed w(m/sec);

$$\rho N_{z0} = 4.3w^2 \text{ for } w > 6 \text{ m/sec} \tag{3-29}$$

For most studies, Reynolds' analogy between momentum and diffusive transport is assumed, i.e.,

$$K_{z0} = N_{z0}$$

In the case of a stably stratified ocean (the density increasing with depth) the mean buoyancy field acts to suppress the generation of turbulence. The criteria used to determine the fluid conditions for which turbulence is completely suppressed is based on the Richardson number

$$R_i = \frac{g}{\rho}\frac{\partial\rho/\partial z}{(\partial u/\partial z)^2} \tag{3-30}$$

where ρ is the density of the fluid, u is the current velocity and g the acceleration of gravity in the vertical z direction. The Richardson

number represents the ratio of the rate of suppression of turbulence by the buoyancy field to the rate of production by the action of Reynolds stresses. If R_i is greater than some critical Richardson number, Ri_{crit}, (Ri_{crit} of order .15) then the theory of Ellison[11] indicates that the diffusivity K_z should vanish.

Sundaram and Rehm[12] in their studies of lake thermoclines have recently adopted the following empirical expression to determine the effect of stability on the vertical diffusivity of heat

$$K_H/K_{H_0} = (1 + \sigma_1 R_i)^{-1} \tag{3-31}$$

where σ_1 is a constant found to be approximately .6. In Sundaram and Rehm's analysis, the velocity gradient $\partial u/\partial z$ is assumed to be determined by the expression for a constant stress turbulent layer

$$\partial u/\partial z = u^*/\kappa z \tag{3-32}$$

where κ is Karman's constant = .4 and $u^* = \sqrt{\tau_s/\rho}$, is the friction velocity with τ_s being the surface shear stress. Equations (3-31) and (3-32) were shown to adequately represent both the field measurements in Cayuga Lake[13] and the laboratory measurements in a stratified channel flow (see Fig. 4). An overall empirical expression for the vertical diffusivity coefficient K_z can be obtained by combining Eqs. (3-29) - (3-32), i.e.,

$$K_z = \frac{4.3w^2}{\left[1 + .1\,\dfrac{gz^2\partial\rho/\partial z}{\rho u^{*2}}\right]} \tag{3-33}$$

with w in m/sec. For typical values of the stratified density gradient $\frac{1}{\rho}\frac{\partial\rho}{\partial z}$ ($\approx 10^{-7}$g cm^{-4}) and shear velocity u^* ($\approx$ 33 cm/sec), encountered in the ocean, it is found from Eq. (3-33) that the vertical diffusivity K_z is reduced by a factor of one-half by a depth $z \approx 100$ meters (near the top of the ocean thermocline). At a depth of 300 meters, the magnitude of K_z is only one tenth of its value near the surface, thus demonstrating that a density gradient as small as 10^{-7} g cm^{-4} can virtually eliminate vertical transport in the thermocline.

Ozmidov[14] reports a striking observation of this effect in his experiments of the diffusion of patches of dye (rhodamine C) in the Black Sea during the fall of 1964:

> "In its initial phase, the diffusion of the dye from an instantaneous point source is symmetric in all directions. But after the patch reaches dimensions of several meters it begins to "flatten up" sharply, and whereas the increase of the patch along the vertical has practically stopped, the patch continues to increase rapidly along the horizontal directions. After several hours had elapsed, the patch reached horizontal dimensions of the order of 1 km, whereas its vertical extent did not usually exceed 15 to 20 m (in this case the layer of discontinuity was located at a depth of ~60m)."

Ozmidov[14] has proposed a simple theory for turbulent diffusion in the ocean, which seeks to quantitatively model the anisotropic effects noted in his ocean dye diffusion experiment. His arguments are as follows:

If the energy supply of the ocean is sufficiently large, an interval of scales from ℓ_0 to ℓ_{cr} will exist in which the turbulence is isotropic. The lower bound ℓ_0 is equivalent to the smallest dissipation length scale of the inertial subrange, i.e.,

$$\ell_0 = (\nu^3/\varepsilon)^{1/4} \tag{3-34}$$

where ν is the kinematic viscosity of sea water and ε is the rate of turbulent energy dissipation. The upper bound for which the turbulence will be isotropic is characterized by the eddy size ℓ_{cr} for which the stratification density barrier is just overcome (ℓ_{cr} is related to the Richardson number based on ε), i.e.,

$$\ell_{cr} = \left[\frac{\rho\varepsilon^{2/3}}{g(\partial\rho/\partial z)}\right]^{3/4} \tag{3-35}$$

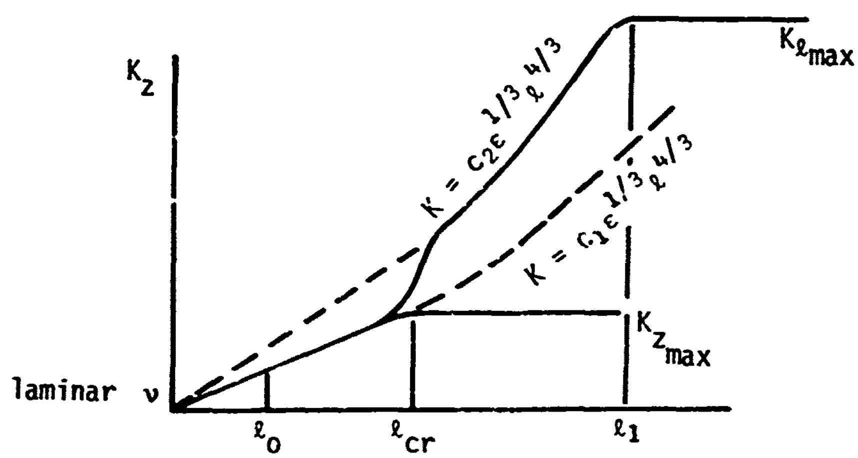
In the interval $\ell_0 < \ell < \ell_{cr}$ the diffusion of impurities from a instantaneous point source is identical for all directions and a cloud of diffused matter will have a spherical shape. Eddies with dimensions greater than ℓ_{cr} can no longer take part in the vertical diffusion transfer, thus limiting the

effective vertical diffusion coefficient K_z. The limiting value of K_z can be estimated from the 4/3 power law for isotropic turbulence

$$K_{z_{max}} = C_1 \varepsilon^{1/3} \ell_{cr}^{4/3} = \frac{C_1 \rho \varepsilon}{g(\partial\rho/\partial z)} \quad (3\text{-}36)$$

where C_1 is a constant approximately equal to .1.

Ozmidov argues that in the range of scales from ℓ_0 to ℓ_{cr} in which the turbulence is isotropic the values of the vertical diffusion coefficient K_z and of the horizontal coefficient K_ℓ are equal (and are given by the 4/3 law). However, for phenomena with scales greater than ℓ_{cr}, K_ℓ and K_z become markedly different as illustrated in the sketch below.



For scales $\ell > \ell_{cr}$, K_z reaches its maximum value $K_{z_{max}}$ (given by Eq. (3-36)) determined by buoyancy. The horizontal coefficient, K_ℓ, jumps to a new two dimensional 4/3 law curve (with constant C_2) and continues to increase with ℓ up to some maximum scale ℓ_1 which may be of the order of several kilometers. Sample calculations for typical values of $\partial\rho/\partial z$ and ε in the ocean, yield limiting values of $K_{z_{max}}$ in the range of 10 to 100 cm^2/sec and values of ℓ_{cr} in the range of 1 to 10 meters[14]. These numbers confirm the relatively small extent of vertical diffusion in a stratified ocean.

One essential feature pointed out by this model is that while far field diffusion effects may well be given by the 4/3 law [e.g., Eq. (3-24)] for horizontal diffusion and the corrected Sverdrup law [Eq. (3-32)] for vertical diffusion, these models are not adequate in the near field where diffusion changes from isotropic to anisotropic behavior.

To estimate the effect of the near field behavior on the far field solution, it is possible to seek alternate solutions of the two dimensional diffusion equation [Eq. (3-15)], assuming that the initial field concentration and size are prescribed as distributed initial conditions (rather than point source initial conditions). Appropriate near field solutions for a collapsing submarine wake have recently been set forth by Ko[1] and modifications presented in Section 2. These solutions would provide the needed initial wake dimensions and initial concentration levels.

The reformulated two dimensional (vertical transport neglected) diffusion equation and boundary conditions can be written as

$$U \frac{\partial \overline{c}}{\partial x} = K_y \frac{\partial^2 \overline{c}}{\partial y^2} - \frac{\overline{c}}{\tau_c} \tag{3-37a}$$

with boundary conditions

$$\overline{c} = \overline{c}_0 \text{ at } x = 0 \underset{\text{(virtual origin)}}{} \qquad -\frac{\ell_0}{2} < y \leq \frac{\ell_0}{2} \tag{3-37b}$$

$$\overline{c} \to 0 \text{ as } x \to \infty, \ \overline{c} \to 0 \text{ as } y \to \pm \infty \text{ for all } x$$

where ℓ_0 represents the initial width of the diffusion layer at the joining point with the near field solution.

The solution of Eq. (3-37) will follow the previous analysis of Brooks[15] for diffusion of sewage effluents in an ocean current and the thermal discharge analysis of Sundaram et. al.[16].

Equation (3-37) can be converted to a simple diffusion equation without a decay term by the transformation

$$\overline{c} = \tilde{c} \exp\left[- x/(U\tau_c)\right] \tag{3-38}$$

Then Eq. (3-37) reduces to the form

$$U \frac{\partial \tilde{c}}{\partial x} = K_y(x) \frac{\partial^2 \tilde{c}}{\partial y^2} \tag{3-39}$$

It will be assumed as in Eq. (3-19b) that the horizontal diffusivity is related to the characteristic length $\ell(x)$ by the relation

$$K_y = a\ell^n(x)$$

and (3-40)

$$\ell(x) = b\sigma(x)$$

where σ is the standard deviation of the concentration distribution defined by

$$\sigma(x)^2 = \frac{\int_{-\infty}^{\infty} y^2\tilde{c}dy}{\int_{-\infty}^{\infty} \tilde{c}dy} = \frac{1}{\tilde{c}_0\ell_0}\int_{-\infty}^{\infty} y^2\tilde{c}dy \qquad (3\text{-}41)$$

If one requires that $\ell = \ell_0$ at $x = 0$, where $\tilde{c}$ is taken to be uniform and equals to $\tilde{c}_0$ in the domain $-\ell_0/2 < y < \ell_0/2$, then $b = 2\sqrt{3} \approx 3.4$.

The solution of Eq. (3-39) could be written down immediately if K_y were a constant, independent of x. However, for K_y being a function of x, a simple transformation of the x coordinate can be used such that

$$\begin{aligned} K_0dx' &= K_y(x)dx \\ &= a\left[\frac{b^2}{\tilde{c}_0\ell_0}\int_{-\infty}^{\infty} y^2\tilde{c}dy\right]^{n/2} dx \end{aligned} \qquad (3\text{-}42)$$

Then Eq. (3-39) reduces to the well known diffusion equation with constant coefficient K_0 (= $a\ell_0^n$), i.e.,

$$U\frac{\partial\tilde{c}}{\partial x'} = K_0\frac{\partial^2\tilde{c}}{\partial y^2} \qquad (3\text{-}43)$$

For the initial conditions given by Eq. (3-37b), the closed form solution of Eq. (3-43) given by Crank[17], is

$$\frac{\tilde{c}}{\tilde{c}_0} = \frac{1}{2}\left\{\operatorname{erf}\frac{2y + \ell_0}{4\sqrt{K_0x'/U}} - \operatorname{erf}\frac{2y - \ell_0}{4\sqrt{K_0x'/U}}\right\} \qquad (3\text{-}44)$$

The integral in Eq. (3-42) which is used to relate the physical x and transformed x' variables, can now be evaluated by employing the error function solution given in Eq. (3-44), i.e.,

$$\frac{\ell^2}{\ell_o^2} = \frac{b^2}{\tilde{c}_o \ell_o^3} \int_{-\infty}^{\infty} y^2 \tilde{c} dy = 1 + 2 \frac{b^2}{\ell_o^2} \frac{K_o x'}{U} \tag{3-45}$$

Hence the transformed axial dimension x' is found in terms of x by integrating Eqs. (3-42) and (3-45),

$$x' = \frac{\ell_o^2 U}{2b^2 K_o} \left\{ \left[1 + (1 - \frac{n}{2}) \frac{2b^2 K_o}{\ell_o^2 U} x \right]^{2/(2-n)} - 1 \right\} \tag{3-46}$$

For large values of x, far from the source, and for n = 4/3 (Richardson law)

$$x' \sim x^3 \qquad x/\ell_o \gg 1$$

Under the same circumstances, the concentration solution, Eq. (3-44), takes the limiting form,

$$\tilde{c} \sim (x')^{-1/2} \qquad x/\ell_o \gg 1$$

Hence, the point source behavior, given by Eqs. (3-21) and (3-24)

$$\tilde{c} \sim (x)^{-3/2}$$

is recovered from the solution, Eq. (3-44), for diffusion from a finite near wake source.

In summary, the complete solution for diffusion from a finite source of dimension ℓ_o and concentration $\bar{c}_o$, with decay τ_c is written in the following non-dimensional form

$$\frac{\bar{c}}{\bar{c}_o} = \frac{1}{2} \exp\left[- \tilde{x}/\tilde{L}_D \right] \left\{ \operatorname{erf} \frac{2\tilde{y} + 1}{4\sqrt{\tilde{K}_o \tilde{x}'}} - \operatorname{erf} \frac{2\tilde{y} - 1}{4\sqrt{\tilde{K}_o \tilde{x}'}} \right\} \tag{3-47}$$

$$\tilde{x}' = \frac{x'}{\ell_0} = \frac{1}{2b^2\tilde{K}_0}\left\{\left[1 + (1 - \frac{n}{2})\, 2b^2\tilde{K}_0\tilde{x}\right]^{2/2-n} - 1\right\}$$

where $\tilde{x} = x/\ell_0$, $\tilde{y} = y/\ell_0$, $L_D = U\tau_c/\ell_0$, and $\tilde{K}_0 = \frac{K_0}{\ell_0 U} = \frac{a\ell_0^{n-1}}{U}$

In the final report of this investigation, a parametric study will be performed to determine the influence of the various prominent parameters in Eq. (3-47) on the magnitude and variation of contaminant concentration behind a typical submarine in an ocean environment.

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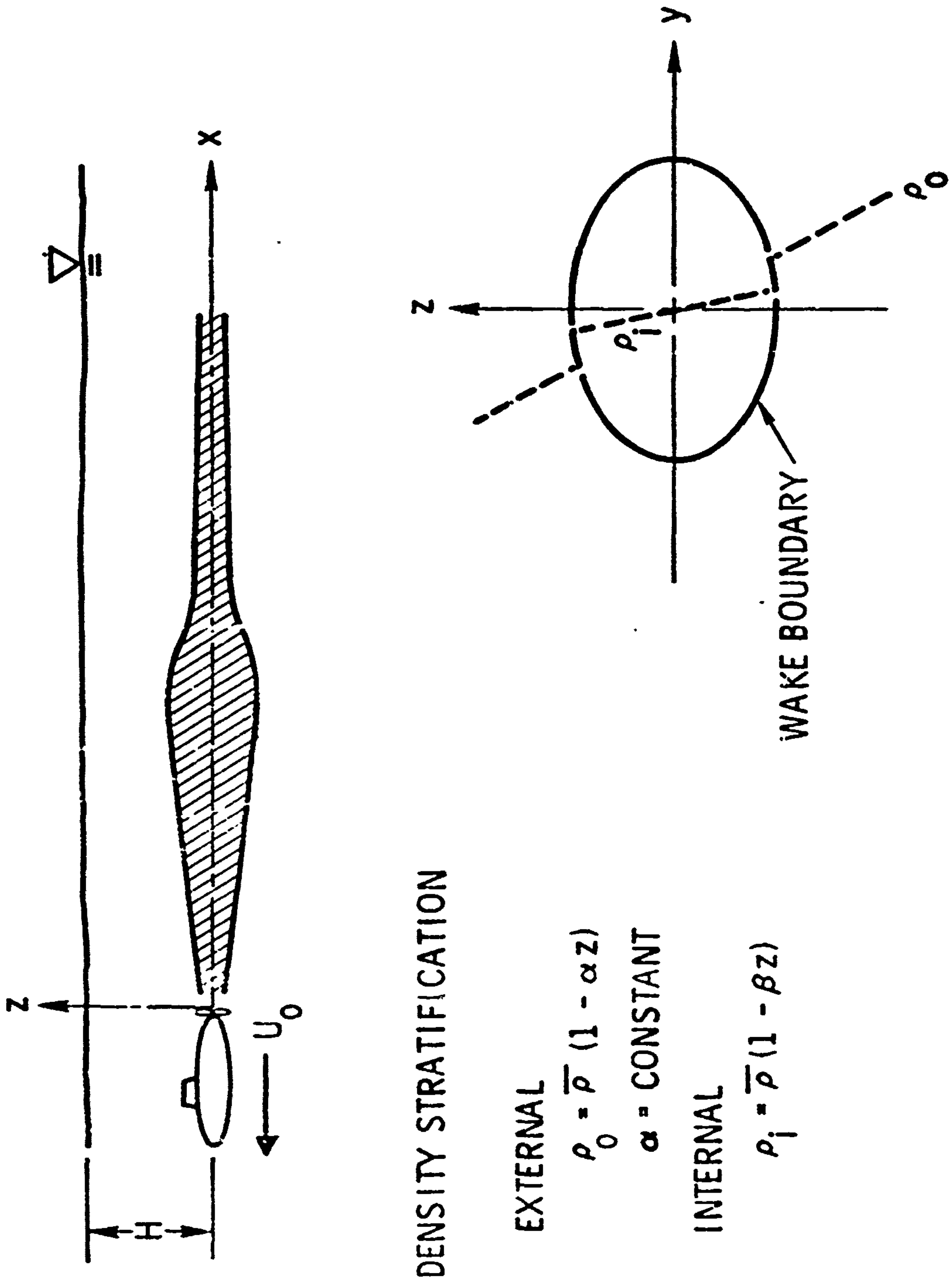

Fig. 1. Coordinates and Density Field

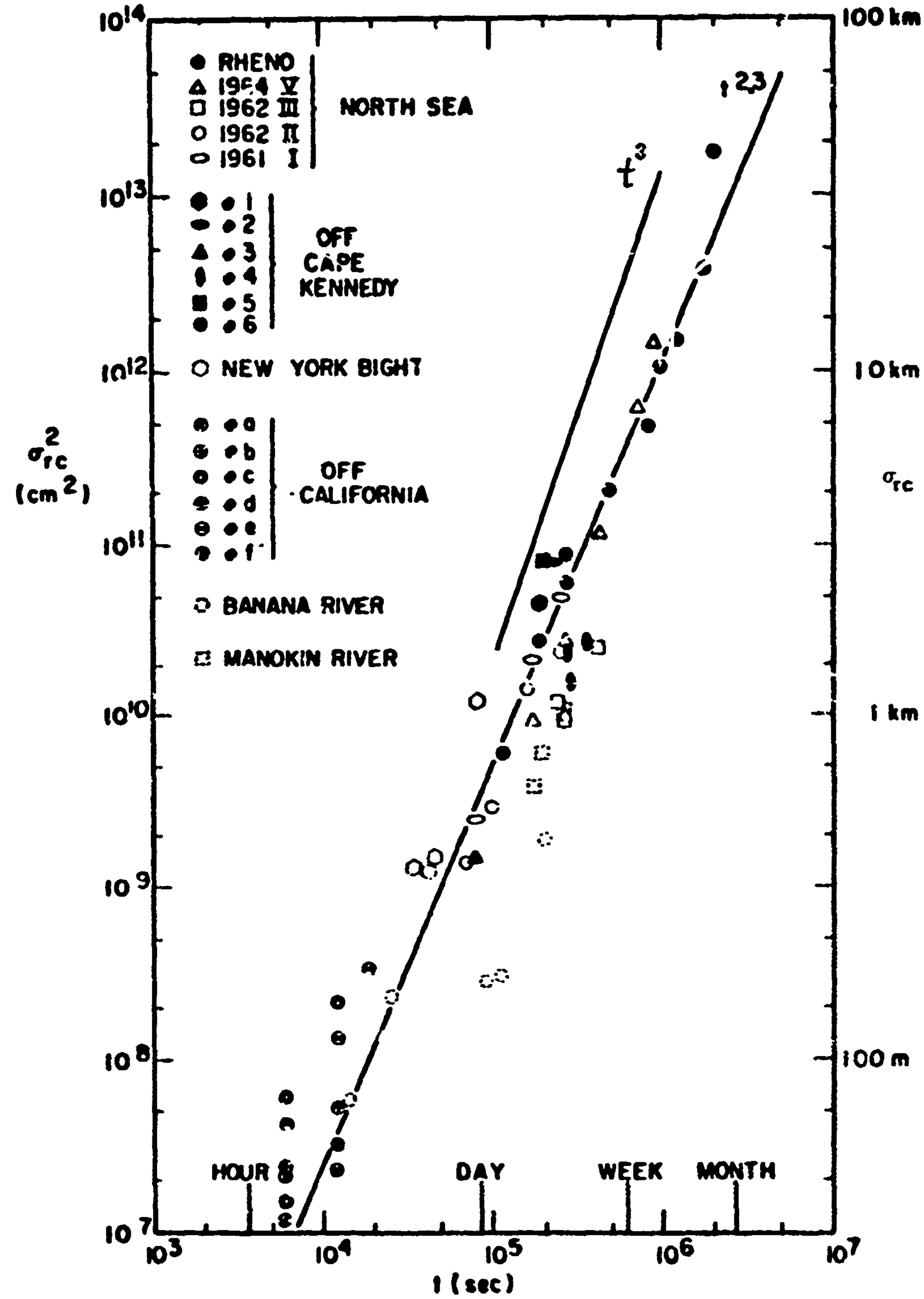

Fig. 2. Variance, σ_{rc}^2, of Dye Distribution against Diffusion Time (Data from 1961 Onwards) (after Okubo, 1968, Ref. 7)

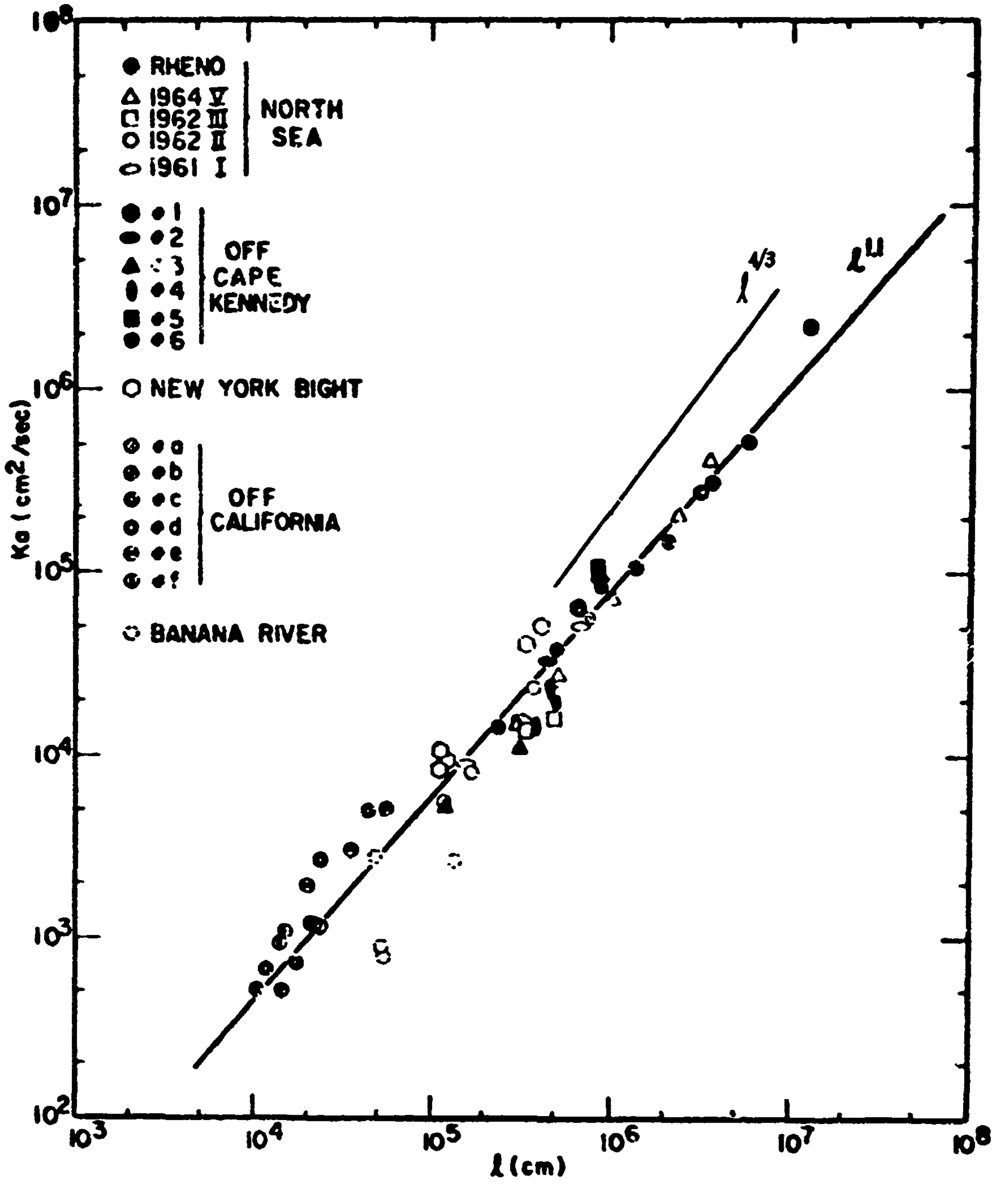

Fig. 3. Apparent Diffusivity, K_a, against Scale of Diffusion, $\ell = 3\sigma_{rc}$ (Data from 1961 Onwards) (after Okubo, 1968, Ref. 7)

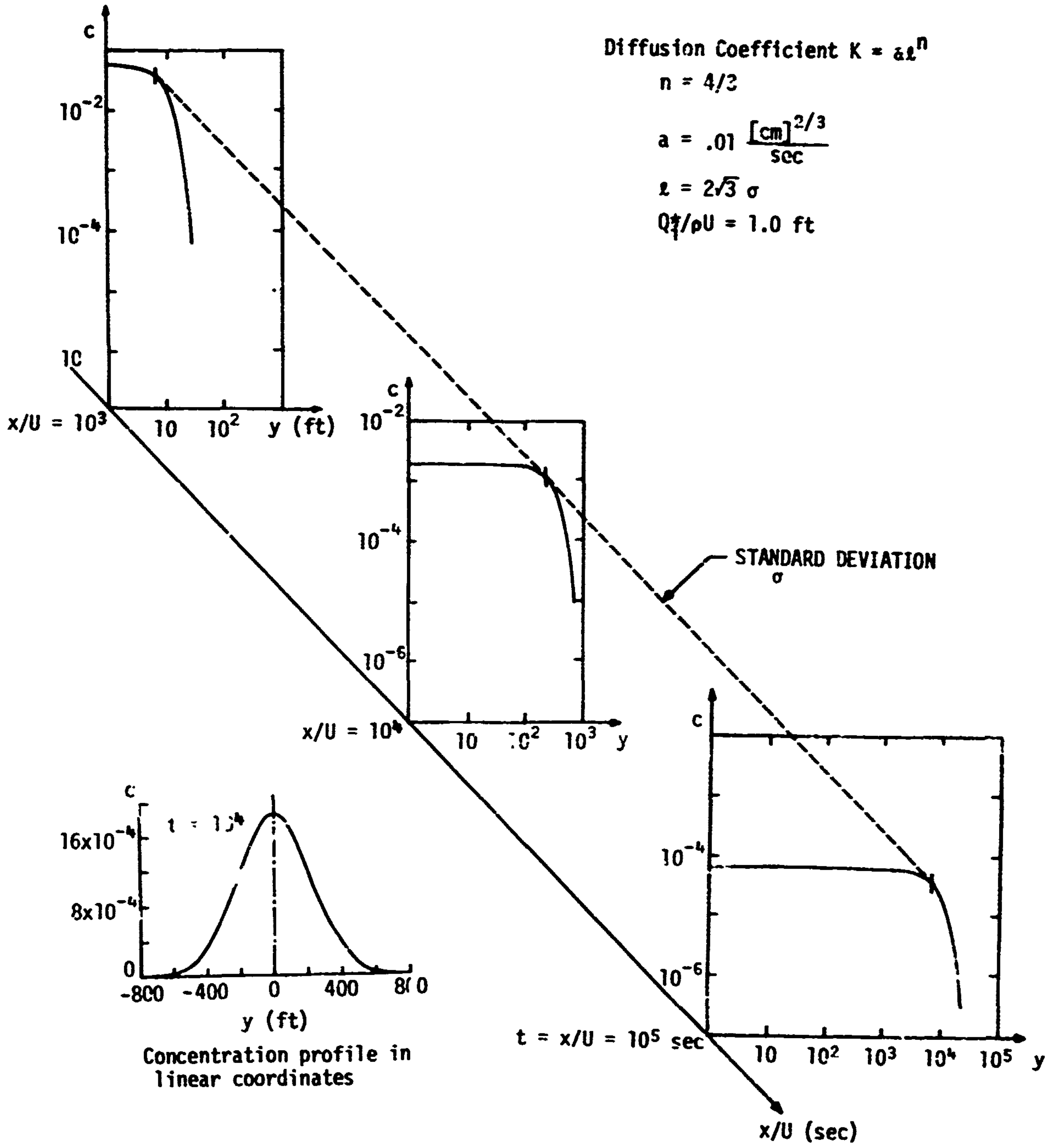

Fig. 4a. Concentration Profiles c Behind a Point Source Moving at a Velocity U in an Ocean Environment, Eq. (3-21)

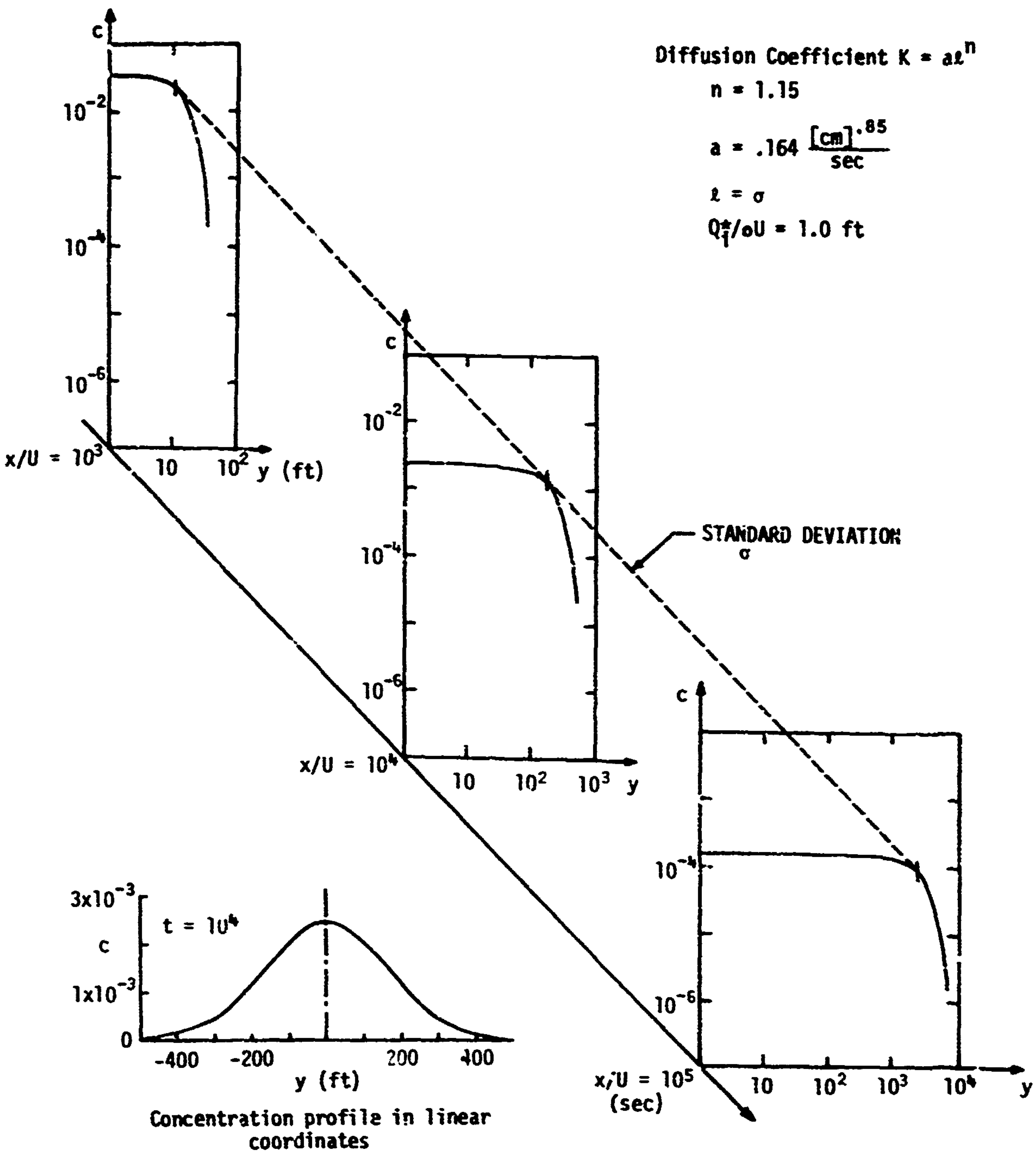

Fig. 4b. Concentration Profiles c behind a Point Source Moving at a Velocity U in an Ocean Environment, Eq. (3-21)

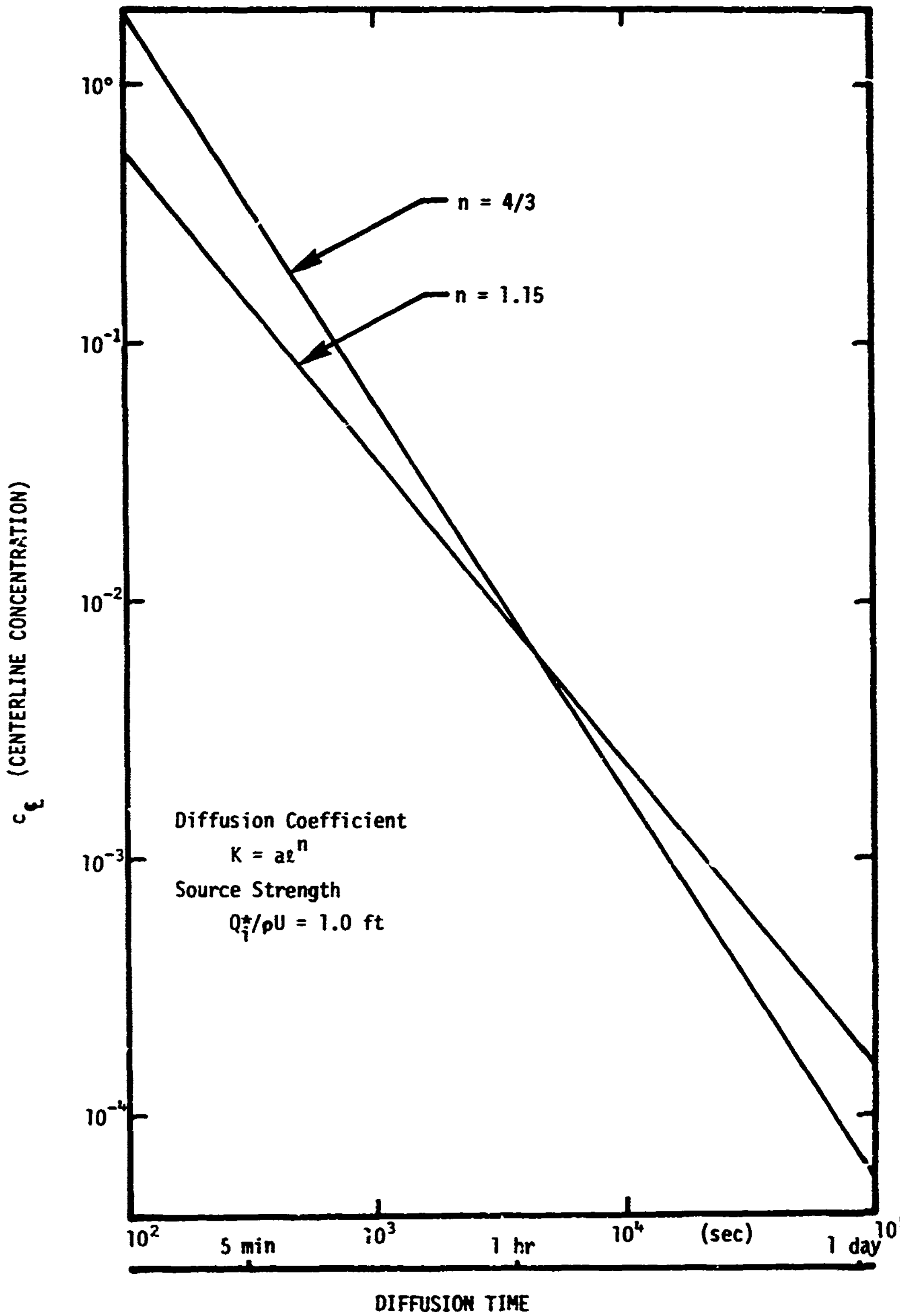

Fig. 4c. Centerline Concentration Behind a Point Source Moving at a Velocity U in an Ocean Environment, Eq. (3-21) ($y = 0$)

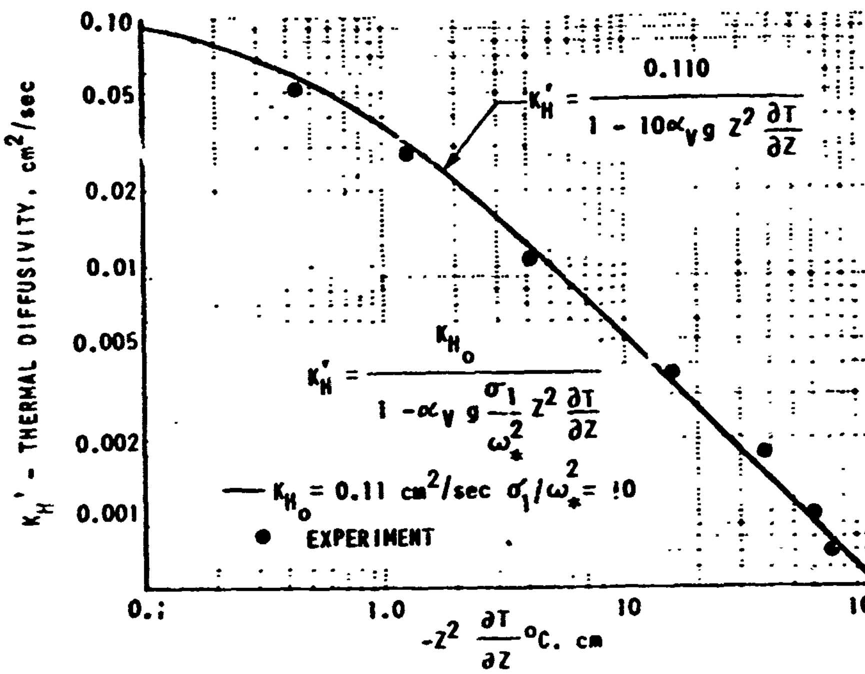

Fig. 5a. Thermal Diffusivity from Stratified Flow Experiments in a Water Channel (after Merritt and Rudinger, Ref. 13) Comparison with Theoretical Model, Eq. (3-31) (after Sundaram and Rehm, Ref. 12)

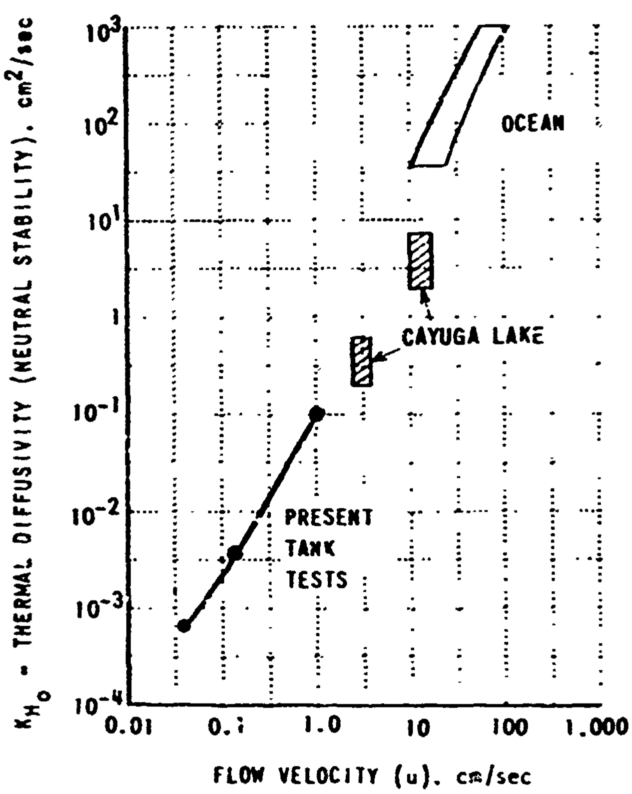

Fig. 5b. Thermal Diffusivity (Neutral Stability) as a Function of Flow Velocity (after Merritt and Rudinger, Ref. 13)